

UPPER BOUND FOR THE WEIL-PETERSSON VOLUMES

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ABSTRACT. An explicit upper bound for the Weil-Petersson volumes of punctured Riemann surfaces is obtained using the combinatorial integration scheme from [4]. It is shown that for a fixed number of punctures n and for genus g increasing,

$$\lim_{g \rightarrow \infty, n \text{ fixed}} \frac{\ln \text{vol}_{WP}(\mathcal{M}_{g,n})}{g \ln g} \leq 2,$$

while this limit is exactly equal to two for $n = 1$.

0. INTRODUCTION

After Wolpert in [7] computed the cohomology of the moduli space of Riemann surfaces as a graded vector space, the question of computing the cohomology ring structure (aka the intersection theory) on the moduli arose. The problem has been intensively studied since then. Witten's paper [6] is a good source for available techniques and ideas. Witten's conjecture, which later became Kontsevich's theorem [2], shows that the intersection numbers satisfy a certain KdV equation.

However, the problem of getting explicit numerical results still remains, as the recursive computations become exceedingly complicated as the genus and number of punctures grow. Carel Faber computed some low-genus intersection numbers in [1] and has obtained numerous results in other papers.

Recently Zograf [8] and Manin and Zograf [3] have obtained quite explicit generating functions for the Weil-Petersson volumes, and computed the asymptotics of the volume growth for genus being fixed and the number of punctures growing to infinity.

In this paper we use a completely different set of tools, namely the decorated Teichmüller theory, to obtain an explicit asymptotic upper bound for the Weil-Petersson volumes for a fixed number of punctures and the genus growing to infinity.

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1. DECORATED TEICHMÜLLER THEORY

Let us recall the notations and relevant constructions. All of these come from Penner's work [4]: this is just a brief summary.

Let $\mathcal{M}_{g,n}$ denote the moduli space of Riemann surfaces of genus g with n punctures — it has complex dimension $3g - 3 + n$. Let ω_{WP} denote the two-form of the Weil-Petersson scalar product on $\mathcal{M}_{g,n}$. It can be extended to a closed current on the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space. Taking its highest power produces a volume form on $\mathcal{M}_{g,n}$, integrating which over $\overline{\mathcal{M}}_{g,n}$ gives the *Weil-Petersson volume* $\text{vol}_{WP}(\mathcal{M}_{g,n})$, which is the principal object of our study. We will only be concerned with hyperbolic punctured surfaces, i.e. the case when $2g + n \geq 3$ and $n > 0$.

An *ideal triangulation* of a punctured surface is a triangulation of the surface with vertices only at punctures. We can straighten an ideal triangulation so that every edge of it is a geodesic arc for the hyperbolic metric on the surface. From Euler characteristics considerations it follows that such a triangulation consists of $V := 4g - 4 + 2n$ triangles, and has $N := 6g - 6 + 3n$ edges.

The *decoration* of a punctured Riemann surface is an addition of a horocycle around each puncture. More rigorously, on the uniformizing hyperbolic plane we take a horocycle around a preimage of a puncture, and consider its projection to the Riemann surface. For a decorated Riemann surface with an ideal geodesic triangulation define the λ -length of arc e of the triangulation to be $\lambda(e) = \sqrt{2e^\delta}$, where δ is the (signed) hyperbolic distance from the point where e intersects the horocycle around one its end to the point of intersection with the horocycle at the other end. It turns out ([5], theorem 3.1) that for a fixed triangulation the λ -lengths establish a homeomorphism of the decorated Teichmüller space $\widehat{\mathcal{T}}_{g,n}$ and $\mathbb{R}_+^{6g-6+3n}$.

An *embedded graph* is a graph embedded in a Riemann surface. Combinatorially it can be represented as a usual graph endowed with a cyclic order of the edges around each vertex. Given an ideal triangulation of a surface, taking its Poincaré dual produces a trivalent embedded graph on the surface, which we denote Γ . The graph Γ has N edges and V vertices. We define $\lambda(e)$, the λ -length of an edge e of the graph, to be the λ -length of its Poincaré dual geodesic arc of the ideal triangulation.

2. MODULI SPACE DESCRIPTION

Our goal is to use Penner's description of the moduli space in the λ -length coordinates to obtain an explicit upper bound of the Weil-Petersson volumes.

For any edge e of the trivalent embedded graph Γ let f_i and g_i be the adjacent edges of the graph at i 's end of e . Then define the associated simplicial coordinate to be

$$X_e := \sum_{i=1}^2 \frac{\lambda(f_i)}{\lambda(e)\lambda(g_i)} + \frac{\lambda(g_i)}{\lambda(e)\lambda(f_i)} - \frac{\lambda(e)}{\lambda(f_i)\lambda(g_i)}.$$

Further, let ρ_i be the sum of simplicial coordinates of the edges in a path around puncture number i , where the path has to go to “the next edge to the left” at each vertex in the cyclic order corresponding to the embedding of the graph. Notice that if we take any edge and start going to “the next edge to the left” from it, we will end up with a loop around some puncture. Since you can “go to the left” in two directions, i.e. since it matters in which way you start going, we have

$$\rho := \sum_{i=1}^n \rho_i = 2 \sum_{i=1}^N X_{e_i} = 2 \sum \frac{\lambda(e)}{\lambda(f)\lambda(g)},$$

where the sum is over all triples of edges having a common vertex, including the possible renamings of e , f and g .

In these notations Penner proves the following result:

Fact 2.1 ([4], 3.2.1 and 3.4.3). *Let ω be a top-dimension differential form on $\mathcal{M}_{g,n}$. Then*

$$(2.1) \quad \int_{\mathcal{M}_{g,n}} \omega = \sum_{[\Gamma]} \frac{1}{\text{Aut } \Gamma} \int_{D(\Gamma)} \pi^*(\omega),$$

where $\pi : \tilde{\mathcal{T}}_{g,n} \rightarrow \mathcal{M}_{g,n}$ is the forgetful projection, $[\Gamma]$ denotes the isomorphism class of an embedded trivalent graph Γ , and $D(\Gamma)$ is the domain in the decorated Teichmüller space given by

$$(2.2) \quad D(\Gamma) = \{\rho_i = 1; X_e > 0 | \forall e \in \Gamma, \forall i = 1 \dots n\}$$

in the λ -coordinates corresponding to the embedded graph Γ .

3. WEIL-PETERSSON VOLUME FORM

In further computations for simplicity we drop the λ 's and simply write e for $\lambda(e)$, if no confusion is possible.

In λ -length coordinates on the moduli the Weil-Petersson two-form is given by ([4], theorem A.2)

$$(3.1) \quad \omega_{WP} = -2 \sum_{v \in \Gamma; e, f, g \ni v} \frac{de}{e} \wedge \frac{df}{f} + \frac{df}{f} \wedge \frac{dg}{g} + \frac{dg}{g} \wedge \frac{de}{e}.$$

The Weil-Petersson volume form is the $3g-3+n$'s external power of w_{WP} . In general, letting I be a multi-index, and denoting the exclusion of factors by a hat, it would be the sum

$$(3.2) \quad \omega_{WP}^{\wedge(3g-3+n)} = \sum_{|I|=n} a_I \prod d \ln \lambda_1 \wedge \dots \wedge \widehat{d \ln \lambda_I} \wedge \dots \wedge d \ln \lambda_N.$$

Proposition 3.1. *In the above notations, $|a_I| \leq 2^N$.*

Proof. Use the expression (3.1) for the two-form to straightforwardly take an exterior power and compute the total number of summands of one kind (with fixed I). Suppose the product contains some $d \ln e$; then it must come in pair with one of the adjoining edges — let it be f_i in the above notations, so there are four choices. Then g_i must come (if it is in the sum) with one of the two edges at its other end, and then the other edge at its other end must come with still another, and so on. Thus we have one factor of four, and many factors of two. Each new factor of four appears if we encounter an edge in I , so the total number of summands of one kind is at most $2^{|I|} 2^{3g-3+n} = 2^{3g-3+2n}$. Recalling the -2 in the two-form, $|a_I| \leq 2^{3g-3+2n} \cdot 2^{3g-3+n} = 2^N$. \square

For the case of one puncture Penner explicitly computes the Weil-Petersson volume form to be ([4], theorem 6.1.2)

$$(3.3) \quad \omega_{WP}^{\wedge(3g-2)} = \pm 2^{4g-2} \sum_{i=1}^N (-1)^i d \ln \lambda_1 \wedge \dots \wedge \widehat{d \ln \lambda_i} \wedge \dots \wedge d \ln \lambda_N.$$

Thus we are dealing with a form which is singular when the λ coordinates approach zero. However, this is not a problem:

Proposition 3.2. *In the domain of integration $D(\Gamma)$ (formula 2.1) for any edge $e \in \Gamma$ we have $\lambda(e) > 4$ in D .*

Proof. We can construct at most two paths around punctures “going to the left” including the edge e — a path can be constructed by deciding at which end of e we start building it. Let one of these paths go through edges f_1, e, f_2 , and the other — through g_1, e, g_2 ; let ρ_i and ρ_j be the sums of the simplicial coordinates of the edges in these paths. Then using the formula for ρ in terms of α -lengths of sectors ([4], lemma 3.4.2) we see that

$$\rho_i > \frac{2}{e} \left(\frac{g_1}{f_1} + \frac{g_2}{f_2} \right) \quad \text{and} \quad \rho_j > \frac{2}{e} \left(\frac{f_1}{g_1} + \frac{f_2}{g_2} \right).$$

Since $\rho_i = \rho_j = 1$ in the domain D , we get

$$2 > \frac{2}{e} \left(\frac{f_1}{g_1} + \frac{g_1}{f_1} + \frac{f_2}{g_2} + \frac{g_2}{f_2} \right) > \frac{8}{e},$$

so that $e > 4$. If there were only one path going through e , i.e. if $i = j$, we would have $1 = \rho_i > \frac{8}{e}$, and thus $e > 8$, which is even better. \square

Thus in the domain of integration $D(\Gamma)$ the λ -lengths are bounded below. However, D can have limit points at infinities of λ -lengths, and since the integral of $d \ln x$ does not converge at infinity, we have to deal with this problem in great detail, using the triangle inequality.

4. TRIANGLE INEQUALITY

The problem with the decorated Teichmüller theory is that the domain of integration $D(\Gamma)$ cannot be simply described in λ -lengths. Our success will come from the following observation:

Theorem 4.1 (Triangle inequality: [5], lemma 5.2). *Let e , f , and g be three edges of the graph Γ having a common vertex. Then in the domain of integration $D(\Gamma)$ the triangle inequality between them holds:*

$$(4.1) \quad e \leq f + g.$$

Proof. Assume for contradiction that $e > f + g$. Note that

$$(4.2) \quad \frac{g}{f} + \frac{f}{g} - \frac{e^2}{fg} < -2 \iff e > f + g$$

by clearing the denominators and extracting full squares. Similarly

$$(4.3) \quad \frac{e}{f} + \frac{f}{e} - \frac{g^2}{ef} > 2 \iff g < |e - f|,$$

and both inequalities hold if $e > f + g$.

Denote by f_1 and g_1 the edges at the other end of e . From above it then follows that

$$0 < eX_e = \frac{g_1}{f_1} + \frac{f_1}{g_1} - \frac{e^2}{f_1g_1} + \frac{g}{f} + \frac{f}{g} - \frac{e^2}{fg} < \frac{g_1}{f_1} + \frac{f_1}{g_1} - \frac{e^2}{f_1g_1} - 2$$

using inequality (4.2). But this is just the inequality (4.3) for edges e , f_1 and g_1 , and thus $e < |f_1 - g_1|$. Suppose $f_1 > g_1$; then it follows that $f_1 > g_1 + e$, and we can apply a similar argument to the edges f_2 and g_2 at the other end of f_1 to obtain $f_1 < |f_2 - g_2|$. Continuing this process inductively, and assuming at each step that $f_i > g_i$, we end up with an infinite strictly increasing sequence of edges $e < f_1 < \dots < f_n < \dots$, which is rather hard to achieve on a finite graph. Thus assuming that one triangle inequality among λ -lengths fails, we have arrived at a contradiction. \square

To demonstrate the power of the triangle inequality we prove a simple corollary:

Proposition 4.2. *Let ρ be twice the sum of all simplicial coordinates of all edges in the graph Γ , as before. Denote the minimal edge in the graph by μ . If the triangle inequalities are satisfied, then $\rho < \frac{8V}{\mu}$.*

Proof. For any vertex v denote by e_v , f_v and g_v the edges containing it, with e_v being the maximal among the three. Then using the triangle inequalities we have

$$\frac{\rho}{2} = \sum_v \frac{e_v}{f_v g_v} + \frac{f_v}{e_v g_v} + \frac{g_v}{e_v f_v} \leq \sum_v \frac{f_v + g_v}{f_v g_v} + \frac{1}{f_v} + \frac{1}{g_v} \leq \sum_v \frac{4}{\mu}$$

□

5. STOKE'S THEOREM

In section 3 we obtained an expression for the Weil-Petersson form in terms of λ -lengths. However, this expression has multiple summands, each omitting n variables. In this section we use the Stoke's theorem to combine the integrals of the summands into one integral of the highest degree form over a domain in \mathbb{R}^N .

Proposition 5.1. *Let ω be an $(N - n)$ -form in \mathbb{R}^N . Then*

$$\int_{D(\Gamma)} \omega = \pm \int_{0 \leq \rho_i \leq 1, X_e > 0} d\rho_1 \wedge \dots \wedge d\rho_n \wedge \omega$$

Proof. We apply the Stoke's theorem multiply. Indeed,

$$\begin{aligned} \int_D \omega &= \int_D \rho_1 \omega = \pm \int_{X_e > 0, \rho_2 = \dots = \rho_n = 1, 0 \leq \rho_1 \leq 1} d\rho_1 \wedge \omega = \\ &= \int_{X_e > 0, \rho_2 = \dots = \rho_n = 1, 0 \leq \rho_1 \leq 1} \rho_2 d\rho_1 \wedge \omega = \dots = \pm \int_{0 \leq \rho_i \leq 1, X_e > 0} d\rho_1 \wedge \dots \wedge d\rho_n \wedge \omega. \end{aligned}$$

□

Now we need to deal with the $d\rho_i$ factors. We prove the following

Theorem 5.2. *For I being some set of indices with $|I| = N - n$,*

$$\left| \int_{0 \leq \rho_i \leq 1, X_e > 0} d\rho_1 \wedge \dots \wedge d\rho_n \wedge \prod_{i \in I} \frac{de_i}{e_i} \right| \leq \left| \int_{0 \leq \rho_i \leq 1, X_e > 0} n! \rho^n \prod_{i=1}^N \frac{de_i}{e_i} \right|$$

Proof. Indeed, recall ([4], section 3.3.4) the definition of the α -lengths $\alpha(e, v) := \frac{e}{fg}$ in the usual notations. Lemma 3.4.2 in [4] states that ρ_i

is twice the sum of α -lengths of the sectors it traverses. What matters for us is that it is a sum of some α -lengths. For any α -length we have

$$\frac{\partial \alpha(e, v)}{\partial e} = \frac{1}{e} \alpha(e, v) \quad \text{and} \quad \frac{\partial \alpha(e, v)}{\partial f} = -\frac{1}{f} \alpha(e, v).$$

Thus for $\rho_i = 2 \sum_{j=1}^n \alpha(f_j, v_i)$ we have

$$\left| \frac{\partial \rho_i}{\partial e} \right| = \frac{2}{e} \left| \sum \pm \alpha(f_j, v_j) \right| \leq \frac{2}{e} \sum \alpha(f_j, v_j) = \frac{\rho_i}{e} < \frac{\rho}{e}.$$

Applying this trick to each of $d\rho_i$ yields the theorem. \square

Combining this theorem with the bound on the coefficients of the Weil-Petersson from theorem 3.1, noting that there are $\binom{N}{n} < N^n/n!$ summands in the Weil-Petersson form, and enlarging D to the domain $0 \leq \rho \leq n, X_e > 0 \forall e$, we get the following

Corollary 5.3. *The integral of the Weil-Petersson volume over the domain of integration is bounded by*

$$\int_D \omega_{WP}^{\wedge(3g-3+n)} < 2^N N^n \left| \int_{0 \leq \rho < n, X_e > 0} \rho^n \prod_{i=1}^N \frac{de_i}{e_i} \right|$$

6. TRIANGLE INEQUALITY COMBINATORICS

Now we proceed to show how the triangle inequalities lead to a converging integral as an upper bound for the Weil-Petersson volume. We develop an algorithm for inductive estimation of the integral of the Weil-Petersson volume form over the domain where the triangle inequalities hold.

Definition 6.1. Two edges e and f of the graph are called *linked* if they are adjacent at a vertex v , and the third edge g at v is the minimal of the three (not necessarily strictly). Notice that from the triangle inequalities $e < f + g$ and $f < e + g$ it then follows that $\frac{1}{2} \leq \frac{e}{f} \leq 2$.

Definition 6.2. A *chain* is a sequence of edges in which every two consecutive ones are linked. We will only be interested in maximal chains — the ones which are not a part of any longer chain. Such a chain must either form a loop in the graph, or end by two edges which are minimal at their outer-end vertices.

Definition 6.3. Define a *wheel* to be an ordered sequence of (maximal) chains c_1, \dots, c_m such that for any i there is at least one chain among c_1, \dots, c_{i-1} ending at a vertex inside the chain c_i . In further considerations, we will only be interested in maximal wheels, the ones which cannot be enlarged any further. Basically this means that the ends of all chains in the wheel belong to other chains already included in the wheel, so that there are no edges “sticking out” of the wheel.

For further computations we split the domain of integration into at most 3^V parts by deciding which two edges at each vertex are linked. We then use chains and wheels to introduce some order on the set of edges, and to integrate inductively. In this we will be aided by the following technical lemmas. The notations are as in the definition of linking: e , f and g are the three edges at some vertex, among which g is minimal. We are working in the domain Δ where the triangle inequalities hold, noting that $\Delta \supset D(\Gamma)$ by theorem 4.1.

Lemma 6.4. *For fixed e we have $\int_{\Delta} \frac{df}{f} \leq \ln 4$, where \int_{Δ} denotes integration over the possible values of f within domain Δ for fixed e .*

Proof. Since e and f link, we have $e/2 < f < 2e$. Thus

$$\int_{\Delta} \frac{df}{f} \leq \int_{e/2}^{2e} \frac{df}{f} = \ln 4.$$

□

Corollary 6.5. *Let $e_1 \dots e_m$ be a chain. If we fix (the λ -length of) an edge e_i , then*

$$\int_{\Delta} \prod_{j \neq i} \frac{de_j}{e_j} \leq (\ln 4)^{m-1}$$

Proof. Start from the ends of the chain, and apply the lemma to eliminate edges one by one, coming from the ends towards e_i . □

Lemma 6.6. *If we fix the edge g , then the integral over the linked edges e and f can be estimated as $\int_{\Delta} \frac{dedf}{ef} < 2$*

Proof. Split the integral into two parts depending on whether $e > f$ or $f > e$. The computation for them is identical; if $f < e$, we have

$$\int_{\Delta \cap \{f < e\}} \frac{dedf}{ef} \leq \int_g^{\infty} \frac{df}{f} \int_f^{f+g} \frac{de}{e} = \int_g^{\infty} \frac{df}{f} \ln \left(1 + \frac{g}{f} \right) < \int_g^{\infty} \frac{df}{f} \frac{g}{f} = 1.$$

□

Proposition 6.7. *Starting from two linked edges e and f , construct a wheel consisting of edges $e_1 \dots e_m$, where the list includes e and f themselves. Then for fixed g*

$$\int_{\Delta} \prod_{i=1}^m \frac{de_i}{e_i} \leq (\max(\ln 4, \sqrt{2}))^m$$

Proof. The wheel is a collection of chains $c_1 \dots c_k$. Keep the two edges of c_k in between which some chain c_i ends (existent by definition of a wheel), and integrate the other ones out using corollary 6.5 — by this we pick up a factor of $\ln 4$ for each edge. Then use lemma 6.6 above to integrate out the last two remaining edges of the chain c_k — here we pick up a factor of 2 for two edges, i.e. $\sqrt{2}$ per edge. Performing induction in k finishes the proof. \square

If a wheel were the whole graph, we would be able to estimate the integral using the above proposition. However, if the wheel is not the whole graph, we need to be able to link it to the rest of the graph. Thus we will need the following

Lemma 6.8. *In the usual notations for fixed e we have*

$$\int_{\Delta} \frac{df dg}{fg} < \frac{8}{3}$$

Proof. Indeed, recall that $g > 4$ by proposition 3.2. Thus

$$\int_4^e \frac{dg}{g} \int_{\max(g, e-g)}^g \frac{df}{f} + \int_4^g \frac{dg}{g} \int_e^{e+g} \frac{df}{f} \leq \int_4^{e/2} \frac{dg}{g} \ln \frac{e}{e-g} + \int_4^e \frac{dg}{g} \ln \left(1 + \frac{g}{e}\right)$$

Since $\ln(1+x) < x$ for $x > 0$, for the second summand we have

$$\int_4^e \frac{dg}{g} \ln \left(1 + \frac{g}{e}\right) < \int_4^e \frac{dg}{g} \frac{g}{e} = \frac{e-4}{e} < 1.$$

For the first summand we compute

$$\begin{aligned} \int_4^e \frac{dg}{g} \ln \frac{e}{e-g} &= - \int_4^e \frac{dg}{g} \ln \left(1 - \frac{g}{e}\right) = \int_4^e \frac{dg}{g} \sum_{n=1}^{\infty} \frac{g^n}{ne^n} = \\ &= \sum_{n=1}^{\infty} \frac{1}{ne^n} \int_4^e g^{n-1} dg = \sum_{n=1}^{\infty} \frac{1}{ne^n} \frac{e^n - 4^n}{n} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{5}{3}. \end{aligned}$$

Combining the above estimates, we get the lemma. \square

Theorem 6.9. *Let μ be the minimal edge of Γ , and let $e_1 \dots e_{N-1}$ be all the other edges of the graph. Then for a fixed value of μ and a fixed choice of the two linking edges at each vertex we have*

$$\int_{\Delta} \prod_{i=1}^{N-1} \frac{de_i}{e_i} < \left(\frac{8}{3}\right)^{(N-1)/2}$$

Proof. Construct a wheel w_1 starting from edge μ . If this wheel is not the whole graph, consider a chain $c_{1,1}$ ending on w_1 by at least one end. If it ends on w_1 by the other end also, consider another such chain $c_{1,2}$ and so on, until we either exhaust the graph, or get a chain c_{1,m_1} which has an end not on w_1 . Then construct a wheel w_2 at the other end of the chain c_{1,m_1} . If the union of these two wheels and the chains constructed is still not the whole graph, we repeat the process.

As a result, we decompose the graph into a disjoint union of wheels $w_1 \dots w_k$ and chains $c_{i,j}$ for $i \leq k$ and $j \leq m_i$ such that the chains $c_{i,j}$ have both ends on w_i for $i = k$ or $j < m_i$, and that c_{i,m_i} connects w_i and w_{i+1} for $i < k$. Then use corollary 6.5 to eliminate all edges of chains $c_{k,i}$ except the terminal ones, which end at w_k . Using lemma 6.8, we can then include these terminal edges of $c_{k,i}$'s while implementing the proof of proposition 6.7 — integrating out the edges of w_k one by one. Doing this, we get a factor of $8/3$ for eliminating two edges, instead of the smaller factor of $(\ln 4)^2$, which we were getting originally.

At the last step of integrating over the edges of w_k we use the edge at the end of $c_{k-1,m_{k-1}}$ for lemma 6.6, and thus reduce the problem to $k-1$ wheels. Induction in k then yields the desired result. \square

Now we combine all the above estimates to finally obtain

Theorem 6.10. *In the above notations,*

$$\int_{D(\Gamma)} \omega_{WP}^{\wedge(3g-3+n)} < 2^N 3^V N^n \left(\frac{8}{3}\right)^{(N-1)/2} (2V)^n$$

Proof. Combining the results of corollary 5.3 and proposition 4.2, we see that the integral in question is bounded above by

$$2^N N^n \left| \int \rho^n \frac{d\mu}{\mu} \prod_{i=1}^{N-1} \frac{de_i}{e_i} \right| < 2^N N^n \left| \int \left(\frac{8V}{\mu}\right)^n \frac{d\mu}{\mu} \prod_{i=1}^{N-1} \frac{de_i}{e_i} \right|.$$

Using theorem 6.9, we can integrate out all variables except μ , by acquiring an extra factor of $(8/3)^{(N-1)/2}$. Remembering the factor of 3^V for choosing the minimal edge at each vertex, our final upper bound

becomes

$$2^N 3^V N^n \left(\frac{8}{3}\right)^{(N-1)/2} (8V)^n \int_4^\infty \frac{d\mu}{\mu^{n+1}} = 2^N 3^V N^n \left(\frac{8}{3}\right)^{(N-1)/2} (2V)^n$$

□

Using our explicit knowledge of the Weil-Petersson volume form, in the case of one puncture we get

$$\int_D \omega_{WP}^{\wedge(3g-3+n)} < 2^{4g-2} 3^V (8/3)^{(N-1)/2} N.$$

7. CONCLUSION

For the case of one puncture, combining our estimates with Penner's asymptotic computation of the number of cells in formula 2.1 (being $\frac{(2g)!}{N} (e/6)^{-2g}$), and estimating $1/\text{Aut } \Gamma$ from above by one, we finally get

$$\text{vol}_{WP}(\mathcal{M}_{g,1}) < (2g)! 2^{4g-2} 3^{4g-2} (\ln 4)^{6g-3} \left(\frac{6}{e}\right)^{2g} =: c_1^g(2g)!,$$

which has the same leading order infinity as Penner's lower bound

$$\text{vol}_{WP}(\mathcal{M}_{g,1}) > \left(\frac{8e^2}{9}\right)^{2g} \frac{(2g)!}{2(6g-3)^2} =: c_2^g(2g)!,$$

where c_i are some explicit constants. Thus we have proven that

$$c_2^g(2g)! < \text{vol}_{WP}(\mathcal{M}_{g,1}) < c_1^g(2g)! \quad \text{and} \quad \lim_{g \rightarrow \infty} \frac{\ln \text{vol}_{WP}(\mathcal{M}_{g,1})}{g \ln g} = 2.$$

Intuitively, for more than one puncture the number of graphs should grow with genus in the same way as for one puncture, since all the punctures are far away from the additional handles being added, and their number should not matter.

Rigorously, let $T(g, n)$ denote the set of isomorphism classes of ideal triangulations of a surface of genus g with n punctures. Then we prove

Proposition 7.1. *There is a following upper bound on the number of triangulations:*

$$|T(g, n)| < \frac{N^2}{2} |T(g, n-1)| < \dots < \frac{N^{2n-2}}{2^{n-1}} |T(g, 1)| < \frac{(2g)! N^{2n-3}}{2^{n-1}} \left(\frac{6}{e}\right)^{2g}$$

Proof. For $n > 1$ we construct a relation $\phi \in T(g, n) \times T(g, n-1)$ in the following way. Consider two distinct punctures p_1 and p_2 connected by an edge e of a triangulation $x \in T(g, n)$ — if such did not exist, i.e. if all edges emanating from a puncture went back to the puncture itself, it would not be a triangulation of the surface. Shrinking e to a point, and collapsing triangles on both sides of e into arcs of a triangulations, thus identifying p_1 and p_2 , produces a new triangulation $y \in T(g, n-1)$. We define ϕ to be the set of all pairs (x, y) obtained in such a way.

Now consider some $y \in T(g, n-1)$. Any graph in $x \in T(g, n)$ such that $(x, y) \in \phi$ can be reconstructed from y by “blowing up” a pair of edges emanating from a vertex to triangles. Since there are $(6g-6+3(n-1))(6g-6+3(n-1)-1)/2 < N^2/2$ ways to choose a pair of edges of y , there are at most $N^2/2$ triangulations $x \in T(g, n)$ such that $(x, y) \in \phi$. The argument works for all y , and thus $|T(g, n)| < N^2|T(g, n-1)|/2$. Applying this argument until we decrease the number of punctures to one, and then utilizing Penner’s asymptotic computation for that case finishes the proof. \square

Combining all our estimates, for $g \gg n$ we get

$$\text{vol}_{WP}(\mathcal{M}_{g,n}) < \frac{(2g)!N^{2n-3}}{2^{n-1}} \left(\frac{6}{e}\right)^{2g} 2^N 3^V N^n \left(\frac{8}{3}\right)^{N/2} (2V)^n < C^g (2g)!,$$

where c is any constant greater than $2^{17}3^3/e^2$ (notice that is is independent of n , since the g^n has a lower growth order), and thus

$$\lim_{g \rightarrow \infty, n \text{ fixed}} \frac{\ln \text{vol}_{WP}(\mathcal{M}_{g,n})}{g \ln g} \leq 2.$$

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